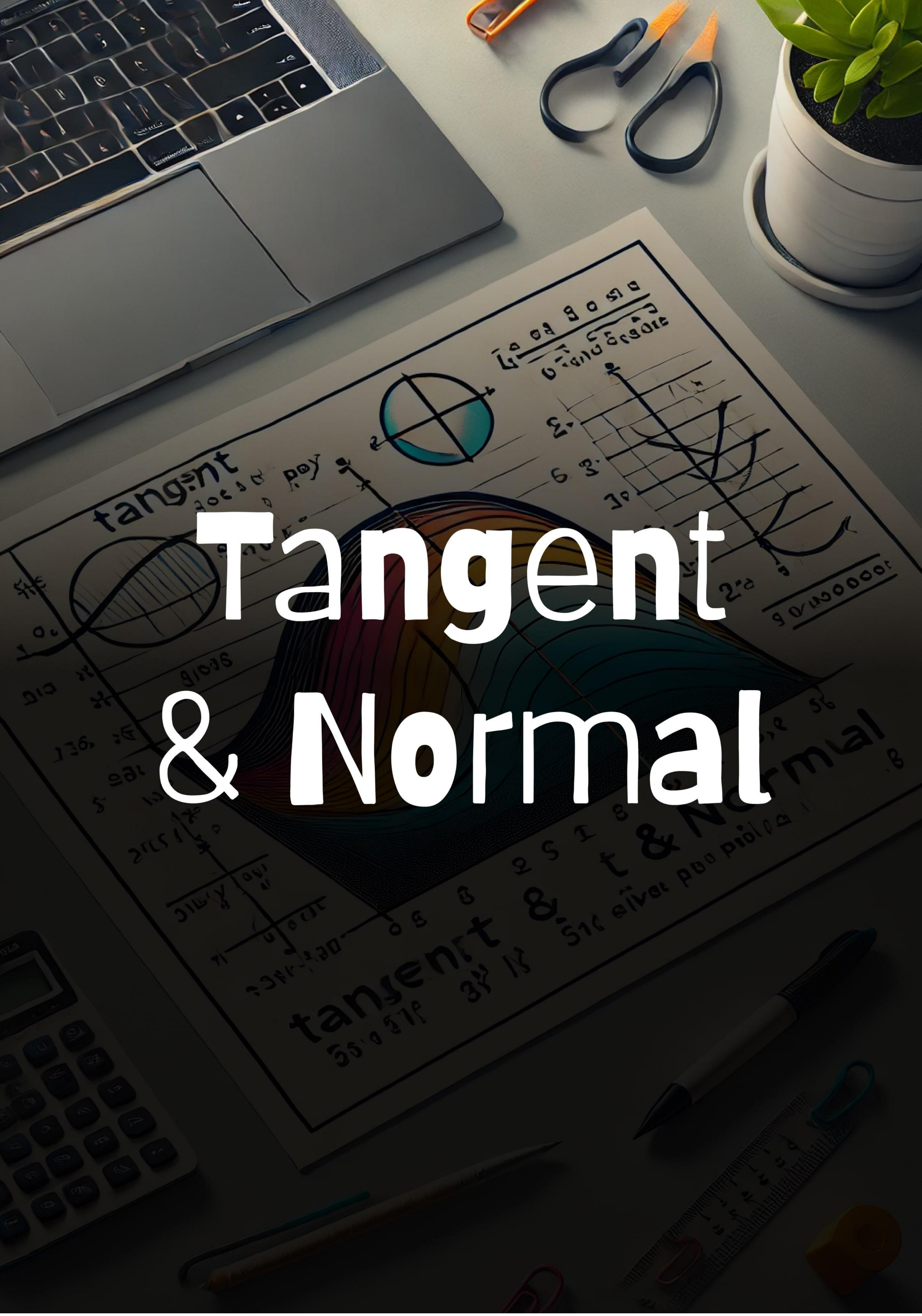
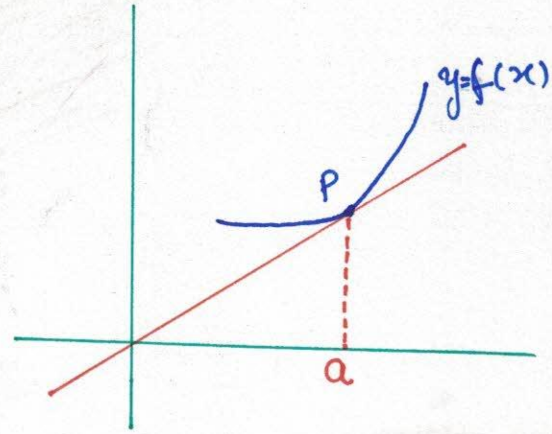


Tangent & Normal

TANGENT & NORMAL



$P = [a, f(a)]$

$m = \left.\frac{dy}{dx}\right|_{x=a} = \left.f'(x)\right|_{x=a} = f'(a)$

eqⁿ of line ⇒

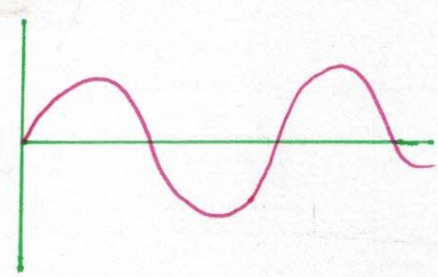
$y - f(a) = f'(a)(x-a)$ if $f'(a)$ exists finitely

if $|f'(a)| \to \infty$

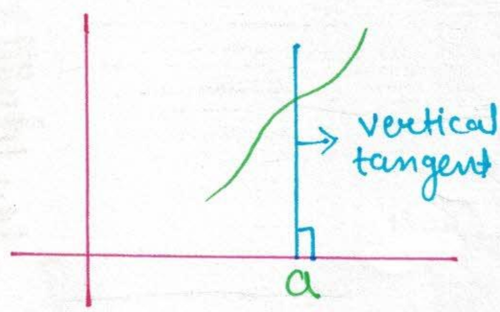
then eq. of tangent ⇒ $x = a$ ⟶ Vertical tangent

In general, a tangent can touch a function more than one time i.e other than Point of Contact

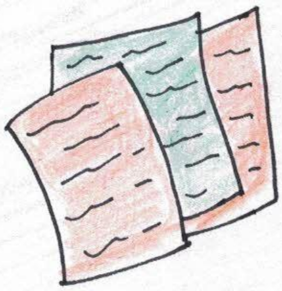
eg $y = \sin x$



In this $y = 1$ will be tangent and will touch infinity and many Points.



Ex: Find the eq. of tangent to the curve $y = e^x + x^2 + 1$ at $x = 1$



$\rightarrow$ Point $\Rightarrow (1, e+2)$

$$\left.\frac{dy}{dx}\right|_{x=1} \Rightarrow \left.(e^x + 2x)\right|_{x=1} = (e+2)$$

$$(y - (e+2)) = (e+2)(x-1)$$

$$y - e - 2 = ex - e + 2x - 2 \Rightarrow y - ex - 2x = 0$$

$$y - x(e+2) = 0 \Rightarrow \boxed{y = x(e+2)}$$

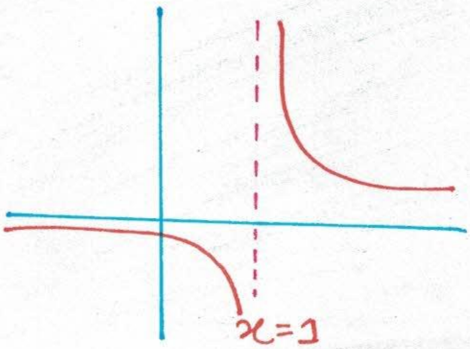
Question

Find eq^n of tangent of $y = \frac{1}{x-1}$

$$\frac{dy}{dx} = m = \frac{d}{dx}(x-1)^{-1} \Rightarrow -1(x-1)^{-2} \times 1 = \frac{1}{(x-1)^2}$$

$$f'(x) = \frac{-1}{(x-1)^2}$$

$$f(x) = \frac{1}{x-1}$$



$x=1$ is an asymptote to $y = \frac{1}{x-1}$
& $y=0$ too.

Question

Find the sum of intercepts made by the tangent line to the axis on the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ at any point (x_1, y_1) to the curve.

$\longrightarrow$ $$\sqrt{x} + \sqrt{y} = \sqrt{a}$$

$$m = \frac{dy}{dx} \Rightarrow \frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0$$

$$\left.\frac{dy}{dx}\right|_{(x_1, y_1)} = \frac{-2\sqrt{y}}{2\sqrt{x}} = -\sqrt{y/x}$$

$$(y-y_1) = -\sqrt{\frac{y}{x_1}}\,(x-x_1)$$

$y=0$

$$-y_1 = -\sqrt{\frac{y_1}{x_1}}\,(x-x_1)$$

$$\sqrt{x_1 y_1} = x - x_1 \Rightarrow x = x_1 + \sqrt{x_1 y_1}$$

$x=0$

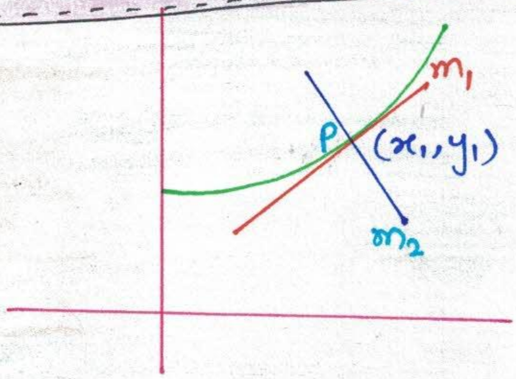
$$y-y_1 = -\sqrt{\frac{y_1}{x_1}}\, x - x_1$$

$$y = y_1 + \sqrt{x_1 y_1}$$

$$x + y = x_1 + y_1 + \sqrt{x_1 y_1} + \sqrt{x_1 y_1} \Rightarrow x_1 + y_1 + 2\sqrt{x_1 y_1}$$

$$\Rightarrow (\sqrt{x_1} + \sqrt{y_1})^2 = (\sqrt{a})^2 = a$$

Eq. of NORMAL OF $Y = F(x)$ CURVE AT PT. (x, y)



m_1 is slope of tangent

m_2 is slope of normal

$$m_1 m_2 = -1$$

$$m_2 = \frac{-1}{\left.\frac{dy}{dx}\right|_{x=P}}$$

Eq $\Rightarrow$ $(y-y_1) = m_2 (x-x_1) \Rightarrow \boxed{y - y_1 = \frac{-1}{\left.\frac{dy}{dx}\right|_{x=P}}(x-x_1)}$

★ if $f'(a) = 0$ then eq. of Normal is $x = a$ vertical line.

Ques Find the eq^n of normal to the curve $y = x^3+3x^2-4$ at $x=1$.

$\longrightarrow$ $x=1$, $y = 1+3-4 = 0$

$(x, y) = (1, 0)$

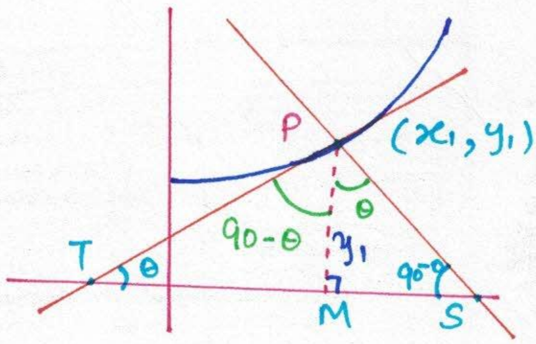
$\frac{dy}{dx} = 3x^2+6x \Rightarrow \left.\frac{dy}{dx}\right|_{x=1} \Rightarrow 3+6 = 9$

Slope $= \frac{-1}{9}$

$(y-0) = \frac{-1}{9}(x-1) \Rightarrow 9y = -x+1$

$\Rightarrow \boxed{x+9y = 1}$

LENGTH OF TANGENT, NORMAL, SUB TANGENT SUB NORMAL ...



PT = length of tangent

PS = length of normal

PM = length of sub tangent

MS = length of sub normal

$\tan\theta = \left.\frac{dy}{dx}\right|_{x=x_1}$ $\quad\quad$ $\sin\theta = \frac{PM}{PT} = \frac{y_1}{PT}$

$PT = y_1 \operatorname{cosec}\theta = |y_1|\sqrt{1+\cot^2\theta}$

$PT = |y_1|\sqrt{1+\left(\frac{1}{dy/dx}\right)^2}$

$$\boxed{\text{length of tangent } PT = |y_1|\sqrt{1+\left(\frac{dx}{dy}\right)^2}}$$

$\frac{PM}{PS} = \sin(90-\theta) = \cos\theta$

$y_1/PS = \cos\theta \Rightarrow PS = y_1\sec\theta = y_1\sqrt{1+\tan^2\theta}$

$$PS = y_1\sqrt{1+(dy/dx)^2}$$

$$\boxed{\text{length of normal} = |y_1|\sqrt{1+(dy/dx)^2}}$$

$$\frac{PM}{MS} = \tan(90-\theta) = \cot\theta$$

$$\frac{y_1}{MS} = \cot\theta \Rightarrow MS = y_1\tan\theta = |y_1|\frac{dy}{dx}$$

$$\boxed{\text{Length of sub normal} = |y_1|\frac{dy}{dx}}$$

$$\frac{PM}{MT} = \tan\theta$$

$$MT = y_1\cot\theta = y_1\frac{dx}{dy}$$

$$\boxed{\text{Length of sub tangent} = |y_1|\frac{dx}{dy}}$$

Find the equation of tangent | normal to the curve $y = b\sin t$, $x = a\cos t$ at $t = \frac{\pi}{4}$ and all length.

$$\frac{dy}{dx} = b\cos t, \quad \frac{dx}{dt} = -a\sin t$$

$$\frac{dy}{dx} = \frac{-b}{a}\cot t = -\frac{b}{a}\cot\frac{\pi}{4} = -\frac{b}{a}$$

$$y = \frac{b}{\sqrt{2}} \qquad x = \frac{a}{\sqrt{2}}$$

$$\boxed{y - \frac{b}{\sqrt{2}} = -\frac{b}{a}\left(x - \frac{a}{\sqrt{2}}\right)}$$

$$\text{length of tangent} = \frac{b}{\sqrt{2}}\sqrt{1+\frac{a^2}{b^2}}$$

Length of normal $= \frac{b}{\sqrt{2}}\sqrt{1+\frac{b^2}{a^2}}$

Length of sub tangent $= \frac{a}{\sqrt{2}}$

Length of sub normal $= -\frac{b^2}{a\sqrt{2}}$

Question

Find eq^n of tangent / normal and all length of curves $x = a\cos^3\theta$, $y = a\sin^{-3}\theta$ at $\theta = \frac{\pi}{4}$.

$\longrightarrow$ $x = a \times \left(\frac{1}{\sqrt{2}}\right)^3 = \frac{a}{2\sqrt{2}}$, $y = \frac{a}{2\sqrt{2}}$

$\frac{dx}{d\theta} = -a \times 3\cos^2\theta \sin\theta$ $\qquad$ $\frac{dy}{d\theta} = a \times 3\sin^2\theta\cos\theta$

$$\frac{dy}{dx} = \frac{\cos\theta \sin^2\theta}{\cos^2\theta \sin\theta} = \tan\theta = 1$$

$$\left(y - \frac{a}{2\sqrt{2}}\right) = \left(x - \frac{a}{2\sqrt{2}}\right) \Rightarrow \boxed{x + y = \frac{a}{\sqrt{2}}} \rightarrow \text{Normal}$$

$\boxed{y = x}$ $\rightarrow$ tangent

Length of tangent $= \frac{a}{2\sqrt{2}}\sqrt{1+1} = \frac{a}{2}$

Length of normal $= \frac{a}{2}$

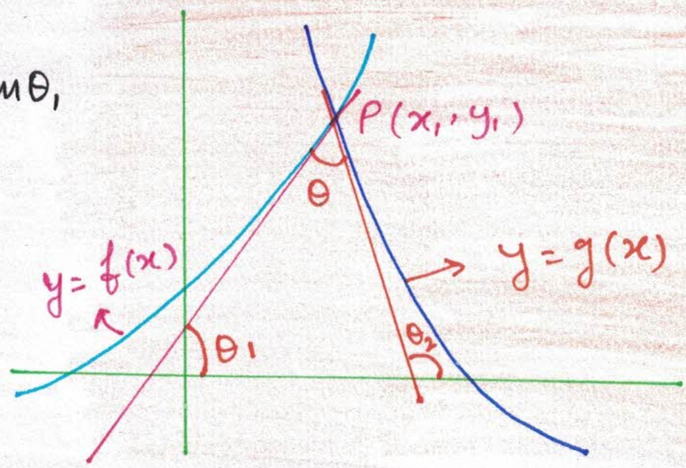
Length of sub tangent $= \frac{a}{2\sqrt{2}}$

Length of sub normal $= \frac{a}{2\sqrt{2}}$

ANGLE B/W TWO CURVE

Angle b/w two curves $y = f(x)$ & $y = g(x)$ is defined as angle b/w tangents at the point of intersection of two curves.

$$\left.\frac{dy}{dx}\right|_{(x_1, y_1)} = f'(x)\Big|_{(x_1,y_1)} = \tan\theta_1$$



$$\left.\frac{dy}{dx}\right|_{(x_2, y_2)} = g'(x)\Big|_{(x_1,y_1)} = \tan\theta_2$$

Angle b/w curve $= |\theta_2 - \theta_1| = \theta$

$$\tan\theta = \left|\frac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_2 \tan\theta_1}\right|$$

$$\tan\theta = \left|\frac{f'(x) - g'(x_1)}{1 + f'(x)\, g'(x)}\right|$$

If the angle b/w curves is 90°, then we say that the curves are orthogonal to each other.

Question Find the angle b/w the curve $y = |x^2 - 1|$ & $y = |x^2 - 3|$

$y = |x^2 - 1|$

$$y = \begin{cases} x^2 - 1, & x \in (-\infty, -1) \cup [1, \infty] \\ 1 - x^2, & x \in (-1, 1) \end{cases}$$

$y = x^2 - 1$

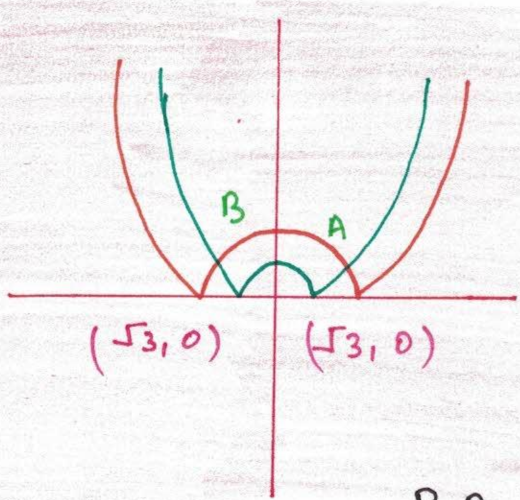
$\frac{dy}{dx} = 2x$

$y = |x^2 - 3|$

$$y = \begin{cases} x^2 - 3, & x \in (-\infty, -\sqrt{3}) \cup (\sqrt{3}, \infty) \\ 3 - x^2, & x \in (-\sqrt{3}, \sqrt{3}) \end{cases}$$

$y = x^2 - 3$

$\frac{dy}{dx} = 2x$



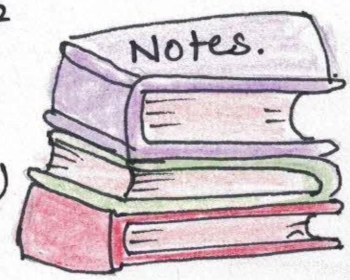
At A, $y = x^2 - 1$

$$y = -x^2 + 3$$

$$\overline{2y = 2 \Rightarrow y = 1}$$

$$1 + 1 = x^2$$

$$x = \pm\sqrt{2}$$



$$\text{P.O.I} = (\sqrt{2}, 1) \;\&\; (-\sqrt{2}, 1)$$

At $(\sqrt{2}, 1)$ A $y = x^2 - 1$

$$\left.\frac{dy}{dx}\right|_{\sqrt{2},1} = 2x = 2\sqrt{2}$$

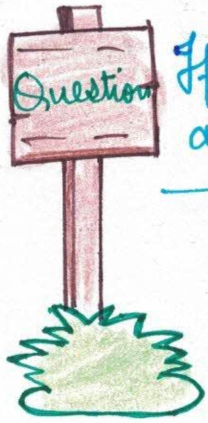
$y = -x^2 + 3$

$$\frac{dy}{dx} = -2x = -2\sqrt{2}$$

$$\tan\theta = \left|\frac{2\sqrt{2} + 2\sqrt{2}}{1 - 8}\right| \Rightarrow \tan\theta = \left|\frac{4\sqrt{2}}{-7}\right|$$

$$\boxed{\theta = \tan^{-1}\left(\frac{4\sqrt{2}}{7}\right)}$$

we will get same angle for point B as curves are Symmetric w.r.t. y axis.



If the curve $y = e^{-2ax}$ and $y = x^2$ intersect orthogonally at $x = 1$ then find a.

$\rightarrow$ $$\left.\frac{dy}{dx}\right|_{x=1} = \tan\theta_1 = -2ae^{-2ax} = -2ae^{-2a}$$

$$\left.\frac{dy}{dx}\right|_{x=1} = \tan\theta_2 = 2x = 2$$

Angle b/w them $= \theta_1 - \theta_2 = 90°$

$$\tan(\theta_1 - \theta_2) = \infty$$

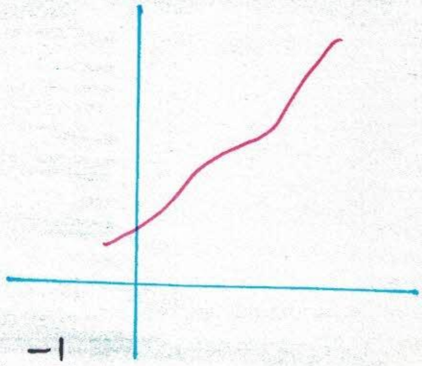
$$\left|\frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2}\right| = \infty$$

$$2x - 2ae^{-2a} = -1$$

$$ae^{-2a} = \frac{1}{4} \Rightarrow 4a = e^{2a}$$ No Possible value of a.

Question If tangent at $P(1,1)$ to the curve $y^2 = x(2-x)^2$ intersect again at Q. find Pt Q.

$$2y = \frac{dy}{dx} = -x \times 2(2-x) + (2-x)^2$$



$$2y\frac{dy}{dx} = 2x^2 - 4x + 4 + x^2 - 4x$$

$$= 3x^2 - 8x + 4$$

$$\frac{dy}{dx} = \frac{3x^2 - 8x + 4}{2\sqrt{x(2-x)^2}} = \frac{3-8+4}{2\sqrt{1 \times 1}} = \frac{-1}{2}$$

$x + 2y = 3 \rightarrow$ eqⁿ of tangent

$$(3-x)^2 = 4x(2-x)^2 \Rightarrow 9 + x^2 - 6x = 4x(4 + x^2 - 4x)$$

$$4x^3 - 17x^2 + 22x - 9 = 0$$

$x = 1$, satisfies

$$(x-1)(4x^2 - 13x + 9) = 0$$

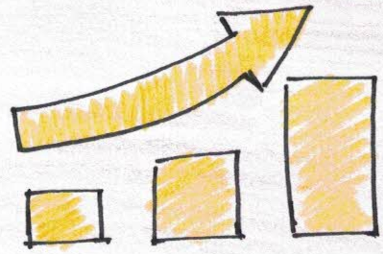
$$(x-1)(x-1)(4x-9) = 0$$

$$x = 1, \frac{9}{4}$$

$$y^2 = \frac{9}{4}\left(2 - \frac{9}{4}\right)^2 = \frac{9}{4} \times \frac{1}{16} \Rightarrow y = \frac{3}{8}$$

$$\boxed{Q = \left(\frac{9}{4}, \frac{3}{8}\right)}$$

Question If tangent at P_1 (other than zero) to the curve $y = x^3$ intersect again at P_2; tangent at P_2 intersect again at P_3 and so on then P.T. x coordinate of $P_1, P_2, P_3 \ldots P_n$ are in G.P. Also find $\frac{\Delta P_1 P_2 P_3}{\Delta P_2 P_3 P_4}$.

Let $P_1 = (x_1, y_1)$ 

$$\frac{dy}{dx} = 3x^2 \Big|_{x=x_1} \Rightarrow 3x_1^2$$

$$y - y_1 = 3x_1^2 (x - x_1) \Rightarrow 3x_1^2 x - 3x_1^3$$

$$y - 3x_1^2 x = y_1 - 3x_1^3 \Rightarrow y = 3x_1^2 x + y_1 - 3x_1^3$$

$$y = 3x_1^2 x + y_1 - 3y_1$$

$$y = 3x_1^2 x - 2y_1$$

$$3x_1^2 x - 2y_1 = x^3 \Rightarrow x^3 - 3x_1^2 x + 2y_1 = 0$$

$$(x - x_1)(x - x_1)(x - 2x_1) = 0$$

$$(x - x_1)^2 (x + 2x_1) = 0 \Rightarrow P_2 = -2x_1$$

$$P_1 (x_1, y_1) \quad P_2 = (-2x_1, -8x_1^3)$$

$$P_3 = (-2x_3, -8x_2^3) = (4x_1, 64x_1^3)$$

$$P_4 = (-8x_1, -512x_1^3)$$

Therefore P_1, P_2 ----- are in G.P.

$$\Delta P_1 P_2 P_3 = \frac{1}{2} \begin{vmatrix} 3x_1 & 63x_1^3 \\ -3x_1 & -9x_1^3 \end{vmatrix} = \frac{x_1^4 \times 3 \times 9}{2} \begin{vmatrix} 1 & 7 \\ -1 & -1 \end{vmatrix}$$

$$= \frac{x_1^4 \times 27 \times 6}{2}$$

$$\Delta P_2 P_3 P_4 = \frac{1}{2} \begin{vmatrix} -6x_1 & -504x_1^3 \\ 6x_1 & +72x_1^3 \end{vmatrix} = \frac{1}{2} \times x_1^4 \times 6 \times 72 \begin{vmatrix} -1 & -7 \\ 1 & 1 \end{vmatrix}$$

$$= \frac{x_1^4 \times 6 \times 72 \times 6}{2}$$

$$\text{Ratio} = \frac{\cancel{27}^{9}}{\cancel{26} \times 720} = \frac{1}{16} \Rightarrow \boxed{\text{Ratio} = \frac{1}{16}}$$

LOCAL PROBLEMS

Question

If tangent at Pt to the curve $x^{2/3}+y^{2/3}=a^{2/3}$ intersect at P & Q will coordinate axes then find locus of mid Pt of PQ.

Question

If tangent are drawn from origin to the curve $y=\sin x$, then find locus of pt. of contact.

$\rightarrow \quad x=a\cos^3\theta \qquad y=a\sin^3\theta$

$$\frac{dx}{d\theta}=-a3\cos^2\theta\sin\theta \qquad \frac{dy}{d\theta}=3a\sin^2\theta\cos\theta$$

$$\frac{dy}{dx}=-\frac{3a\sin^2\theta\cos\theta}{3a\cos^2\theta\sin\theta}=-\tan\theta$$

$$y-a\sin^3\theta=-\tan\theta\,(x-a\cos^3\theta)$$

$$y=0,\quad -a\sin^3\theta=-\tan\theta\,(x-a\cos^3\theta)$$

$$-a\sin^3\theta=-\frac{\sin\theta}{\cos\theta}(x-a\cos^3\theta)$$

$$\Rightarrow -a\sin^2\theta=\frac{-x}{\cos\theta}+a\cos^2\theta$$

$$\frac{x}{\cos\theta}=a \Rightarrow x=a\cos\theta$$

$$y=a\sin\theta$$

Mid Point $=\left(\dfrac{a\cos\theta}{2},\dfrac{a\sin\theta}{2}\right)$

$$\frac{a\cos\theta}{2}=h \qquad \frac{a\sin\theta}{2}=k$$

$$\cos\theta=\frac{2h}{a} \qquad \sin\theta=\frac{2k}{a} \Rightarrow \frac{4h^2}{a^2}+\frac{4k^2}{a^2}=1$$

$$\boxed{x^2+y^2=\frac{a^2}{4}}$$

Answer

$y = \sin x \qquad \{ y_1^2 = \sin x_1^2)$

$\left.\frac{dy}{dx}\right|_{x=1} = \cos x_1$

$(y - y_1) = \cos x_1 (x - x_1)$

$-y_1 = -\cos x_1 \, x_1 \Rightarrow \cos x_1 = \frac{y_1}{x_1}$

$y_1 = \sin x_1$

$y_1^2 + \frac{y_1^2}{x_1^2} = 1 \Rightarrow y_1^2 + x_1^2 y_1^2 = x_1^2$

$x^2 y^2 = x^2 - y^2$

Question Find locus of pt. of contact, tangent at which is ∥ to x axis to the curve $y^2 = 4a\left(x + a\sin\frac{x}{a}\right)$

→ $2y\frac{dy}{dx} = \overset{2}{\cancel{4}}a\left(1 + \cancel{a}\cos\frac{x}{a} \times \frac{1}{\cancel{a}}\right)$

$\frac{dy}{dx} = \frac{2a}{y}\left(1 + \cos\frac{x}{a}\right) \Rightarrow (y - y_1) = \frac{2a}{y}\left(1 + \cos\frac{x}{a}\right) \times (x - x_1)$

$\frac{2a}{y}\left(1 + \cos\frac{x}{a}\right) = 0 \Rightarrow \frac{2a}{y_1}\left(1 + \frac{\cos x_1}{a}\right) = 0$

$\cos\frac{x_1}{a} = -1 \Rightarrow \frac{x_1}{a} = (2n+1)\pi$

$x_1 = (2n+1)\pi a$

$y_1^2 = 4a\left(x_1 + a\sin\frac{(2n+1)\pi\cancel{a}}{\cancel{a}}\right)$

$\boxed{y^2 = 4ax}$

Question Find the angle b/w $y^2 = 4x$ and $y = e^{-x/2}$.

→ $y^2 = 4x \qquad y = e^{-x/2}$

$y^2 = -8\ln y \qquad \ln y = -\frac{x}{2}$

$y^2 + 8\ln y = 0$ $\qquad x = -2\ln y$

$2y\dfrac{dy}{dx} = 4$ $\qquad \dfrac{dy}{dx}\Big|_{x=x_1} = \dfrac{-1}{2}e^{-\frac{x}{2}} = -\dfrac{e^{-\frac{x_1}{2}}}{2} = \dfrac{-y_1}{2}$

$\dfrac{dy}{dx} = \dfrac{2}{y}\Big|_{y=y_1} = \dfrac{2}{y_1}$ $\qquad \boxed{\Rightarrow m_1 m_2 = -1}$ Orthogonal

Question

Find angle b/w $2y^2 = x^3$ and $y^2 = 32x$.

$\rightarrow \quad 4y\dfrac{dy}{dx} = 3x^2$ $\qquad 2y\dfrac{dy}{dx} = 32$

$\dfrac{dy}{dx} = \dfrac{3x^2}{4y} \Rightarrow \dfrac{3}{4}\times\sqrt{2}\sqrt{x}$, $x=0$, $m=0$ $\qquad \dfrac{dy}{dx} = \dfrac{16}{y}$ At $y=0$, $=\infty$

$\boxed{\theta = \dfrac{\pi}{2}}$

P.O.I $\Rightarrow 2\times 32x = x^3$

$2\times 32 = x^2$

$x \neq -8$ $\qquad x = \pm 8$ $\qquad y = \sqrt{32\times x} = |16|$

$(x, y) = (8, 16), (8, -16)$

$\dfrac{dy}{dx}\Big|_{(8,16)} = \dfrac{3\times \cancel{8^2}^{\,4}}{\cancel{4\times 16}} = 3$ $\qquad \dfrac{dy}{dx}\Big|_{(8,16)} = \dfrac{16}{16} = 1$

$\tan\theta = \left|\dfrac{3-1}{1+3}\right| = \dfrac{2}{4}$ $\qquad \dfrac{dy}{dx}\Big|_{(8,-16)} = \dfrac{3\times \cancel{64}^{\,4}}{4\times\cancel{16}} = -3$

$\left|\dfrac{dy}{dx}\right| = \dfrac{16}{-16} = -1$

$\boxed{\theta = \tan^{-1}\left(\dfrac{1}{2}\right)}$ $\qquad \tan\theta = \left|\dfrac{-2}{1+3}\right| = \tan^{-1}\left(\dfrac{1}{2}\right)$

Question Find angle b/w $\frac{x^2}{a^2+k_1} + \frac{y^2}{b^2+k_1} = 1$ and

$$\frac{x^2}{a^2+k_2} + \frac{y^2}{b^2+k_2} = 1$$

$\longrightarrow$ Let P.O.I be (x_1, y_1)

$$\frac{dy}{dx} \Rightarrow \frac{2x}{a^2+k_1} + \frac{2y\frac{dy}{dx}}{b^2+k_2} = 0$$

$$\frac{dy}{dx} = \frac{-x(b^2+k_2)}{y(a^2+k_1)}$$

$$\frac{dy}{dx} = \frac{-x(b^2+k_2)}{y(a^2+k_2)}$$

$$\tan\theta = \left| \frac{\frac{x(b^2+k_2)}{y(a^2+k_2)} - \frac{x(b^2+k_1)}{y(b^2+k_1)}}{1 + \frac{x^2}{y^2}\frac{(b^2+k_1)(b^2+k_2)}{(a^2+k_1)(a^2+k_2)}} \right|$$

$$x = \left(\frac{1}{a^2+k_1} - \frac{1}{a^2+k_2}\right) + y^2$$

$$\left(\frac{1}{b^2+k_1} - \frac{1}{b^2+k_2}\right) = 0$$

$$x^2\left(\frac{k_2-k_1}{(a^2+k_1)(a^2+k_2)}\right) + y^2 \frac{k_2-k_1}{(b^2+k_1)(b^2+k_2)} = 0$$

$$\frac{x^2(k_2-k_1)}{(a^2+k_1)(a^2+k_2)} = \frac{y^2(k_2-k_1)}{(b^2+k_1)(b^2+k_2)}$$

$$\frac{x^2}{y^2} = -\frac{(a^2+k_1)(a^2+k_2)}{(b^2+k_1)(b^2+k_2)}$$

$$\Rightarrow \tan\theta = \left| \frac{-}{1+(-1)} \right|$$

$$\tan\theta = \infty \Rightarrow \boxed{\theta = \frac{\pi}{2}}$$

Question Ques Find the condition for which $ax^2 + by^2 = 1$ and $a_1x^2 + b_1y^2 = 1$ intersect orthogonally.

$\longrightarrow$ $2ax + 2by\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = \frac{-ax}{by} = m_1 \qquad \frac{dy}{dx} = \frac{-a_1x}{b_1y} = m_2$$

$$m_1 m_2 = 1$$

$$\frac{aa_1x_1^2}{bb_1y_1^2} = -1$$

$$ax_1^2 + by_1^2 = 1$$

$$a_1x_1^2 + b_1y_1^2 = 1$$

$$(a-a_1)x_1^2 + (b-b_1)y^2 = 0$$

$$\frac{x_1^2}{y_1^2} = -\frac{(b-b_1)}{a-a_1} \Rightarrow \frac{-aa_1(b-b_1)}{bb_1(a-a_1)} = -1$$

$$\boxed{aa_1(b-b_1) = bb_1(a-a_1)}$$

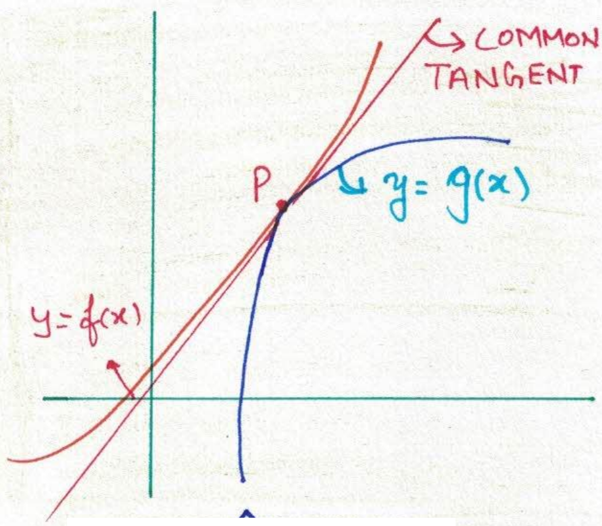
Dividing by $aa_1 bb_1$

$$\frac{b-b_1}{bb_1} = \frac{a-a_1}{aa_1} \Rightarrow \frac{1}{b_1} - \frac{1}{b} = \frac{1}{a_1} - \frac{1}{a}$$

$$\Rightarrow \boxed{\frac{1}{a} - \frac{1}{b} = \frac{1}{a_1} - \frac{1}{b_1}}$$

3 COMMON TANGENT B/w TWO CURVE

$$\frac{d}{dx}(f(x))\Big|_{x=x_1} = \frac{d}{dx}(g(x))\Big|_{x=x_1}$$



Ex:- PT $xy = 4$ and $x^2+y^2 \pm 8$ touch each other at two pts. Also find eq^n of common tangent at that point.

$xy = 4 \qquad x^2+y^2 = 8$

$x = \frac{4}{y} \qquad \frac{16}{y^2} + y^2 = 8$

$16 + y^4 = 8y^2$

$y^4 - 8y^2 + 16 = 0 \qquad y^2 = t$

$t^2 - 8t + 16 = 0$

$\Rightarrow (t-4)^2 = 0 \Rightarrow y^2 = 4$

$y = \pm 2 \qquad x = \pm 2$

$$y + x\frac{dy}{dx} = 0 \qquad\qquad 2x + 2y\frac{dy}{dx} = 0$$

$$\left.\frac{dy}{dx}\right|_{2,2} = \frac{-y}{x} = \frac{-2}{2} = -1 \qquad\qquad \frac{dy}{dx} = \frac{-2x}{2y} = \frac{-2}{2} = -1$$

$$(y-2) = -1(x-2) \qquad\qquad (y+2) = -1(x+2)$$

$$y-2 = -x+2 \qquad\qquad y+2 = -x-2$$

$$\boxed{x+y=4} \qquad\qquad \boxed{x+y=-4}$$

Question $y = \tan^{-1}x$, $A = \left(1, \frac{\pi}{4}\right)$, P move on the curve $P_i = (x_i, y_i)$ $i \in N$ $(i = 1, 2, 3 \ldots n)$ where $y_n = \sum_{r=1}^{n} \tan^{-1}\left(\frac{1}{2r^2}\right)$ and $B = (x, y)$ is a point where $n \to \infty$. find slope of line AB.

$\longrightarrow$ $y_1 = \tan^{-1}\left(\frac{1}{2}\right)$ $\qquad$ $\tan^{-1}\left(\frac{1}{2r^2}\right) = \tan^{-1}\frac{(2)}{(4r^2)}$

$y_2 = \tan^{-1}\left(\frac{1}{4}\right)$ $\qquad$ $\tan^{-1}\left(\frac{2}{1+4r^2-1}\right)$

$y_n = \tan^{-1}(2r-1) - \tan^{-1}(2r-1)$ $\qquad$ $= \tan^{-1}\left(\frac{2}{1+(2r+1)(2r-1)}\right)$

$y_1 = \tan^{-1}(8) - \tan^{-1}(1)$ $\qquad$ $= \tan^{-1}(2r+1) - \tan^{-1}2r$

$y_2 = \tan^{-1}(5) - \tan^{-1}(3)$

$\vdots$

$y_n = \tan^{-1}(2n+1) - \tan^{-1}(2n-1)$ $\qquad$ $y_\infty = \frac{\pi}{4}$ $\quad$ $x = 1$

$\sum = \tan^{-1}(2n+1) - \tan^{-1}(1)$

$$\frac{dy}{dx} = \left.\frac{1}{1+x^2}\right|_{x=1}$$

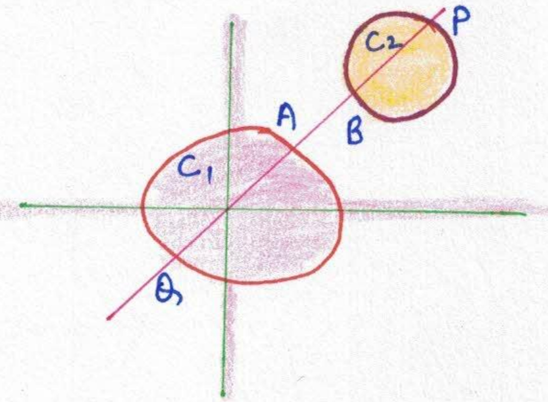
$$\boxed{\Rightarrow \frac{1}{2}}$$

SHORTEST DISTANCE BETWEEN TWO CURVES

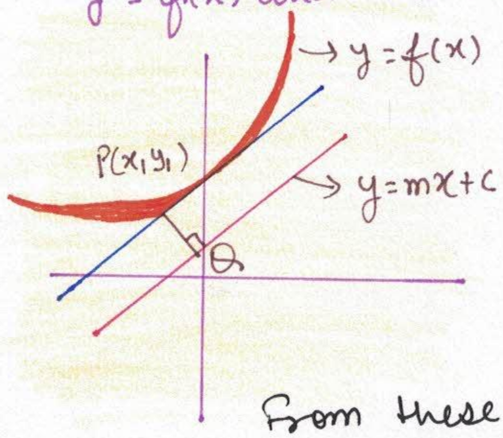
SHORTEST DISTANCE B/W TWO NON-INTERSECTING CURVE

It exists along the common normal to the curves
if longest distance exists, then that is also formed along common normal along the curves.

AB = Shortest dist. b/w C_1 & C_2
PQ = longest dist. b/w C_1 & C_2



Shortest distance b/w a curve $y = f(x)$ and a line $y = mx + c$.



PQ is ⊥ to both $f(x)$ & $mx+c$

So $\left.\frac{dy}{dx}\right|_{(x_1,y_1)} = m$ ---- ①

and P satisfies eq. of curve ---- ②

From these two eq^n we can get (x_1, y_1) & hence shortest distance.

Ex :— Find Shortest distance of curve $y = x^2 + 3x + 2$ from line $y = x - 2$

$\rightarrow \left.\frac{dy}{dx}\right|_{(x_1,y_1)} = 2x + 3$ $\qquad 2x_1 + 3 = 1$

$x_1 = -1$

$y_1 = 1 - 3 + 2 = 0$ $\qquad P = (-1, 0)$

$$\text{Distance} = \frac{|-1-2|}{\sqrt{2}} = \boxed{\frac{3}{\sqrt{2}}}$$

Question Find a pts on the curve $3x^2 - 4y^2 = 72$ which is at min. distance from line $3x + 2y + 1 = 0$

→ Let pt. be (x_1, y_1)

$$\frac{dy}{dx} = 6x - 8y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{3x}{4y} = \frac{3x_1}{4y_1}$$

$$\frac{3x_1}{4y_1} = \frac{-3}{2} \Rightarrow x_1 = -2y_1 \text{ --- } (1)$$

$$3x_1^2 - 4y_1^2 = 72 \Rightarrow 3 \times 4y_1^2 - 4y_1^2 = 72$$

$$8y_1^2 = 72 \Rightarrow y_1 = \pm 3 \quad x_1 = \mp 6$$

$$(x_1, y_1) = (-6, 3), (6, -3)$$

$$\text{Dist.} \Rightarrow \frac{|-18 + 6 + 1|}{\sqrt{13}} = \frac{11}{\sqrt{13}}$$

$$\text{Dist} = \frac{|18 - 6 + 1|}{\sqrt{13}} = \frac{13}{\sqrt{13}}$$

$$\text{Point} = \boxed{-6, 3}$$

Question $C_1 \Rightarrow y = \sqrt{2 - x^2} - P$, $C_2 \Rightarrow xy = 9 - Q$ first Quadrant, if d represent shortest distance b/w P & Q. Find d^2.

→ $y = \sqrt{2 - x^2}$ is a semicircle with centre $(0,0)$

Its normal passes through $(0,0)$

$$\left.\frac{dy}{dx}\right|_P = \frac{-2x}{2\sqrt{2 - x^2}} = \frac{-x}{\sqrt{2 - x^2}}$$

$$m_{normal} = \frac{\sqrt{2 - x^2}}{x}$$

$$x\frac{dy}{dx} + y = 0$$

$$\frac{dy}{dx} = \frac{-y}{x}$$

$$\left. m_{normal} \right|_Q = \frac{x}{y}$$

$$Q = (x_1, y_1)$$

$$m = \frac{x_1}{y_1} \Rightarrow x_1 = my_1 \quad y_1 = \pm\frac{3}{\sqrt{m}} \quad x_1 = \pm\sqrt{m}\,3$$

$$my_1^2 = 9$$

$y = x$ is common normal.

$x^2 + y^2 = 2$

$2y^2 = 2 \qquad y = \pm 1$

$x = \pm 1$

$m = \frac{\sqrt{2}-1}{1} = 1$

$P(x=1 \quad y=1) \qquad Q(3, 3)$

$PQ = \sqrt{8} \qquad d = \sqrt{8} \Rightarrow \boxed{d^2 = 8}$

Question

Find the min. value of $(x_1-x_2)^2 + \left(\frac{x_1^2}{20} - \sqrt{(17-x_2)(x_2-13)}\right)^2$ where $x_1 \in (0, \infty)$ and $x_2 \in (13, 17)$.

$\rightarrow (x_1-x_2)^2 + \left(\underbrace{\frac{x_1^2}{20}}_{y_1} - \underbrace{\sqrt{(17-x_2)(x_2-13)}}_{y_2}\right)^2$

$y_1 = \frac{x_1^2}{20} \Rightarrow x^2 - 20y \rightarrow ①$

$y^2 = (17-x_2)(x_2-13) = 17x_2 - 221 - x_2^2 + 13x_2$

$y^2 = -(x_2^2 - 30x_2 + 221)$

$x^2 + y^2 - 30x + 221 = 0 \rightarrow ②$

we need min^m distance b/w ① & ②

$x^2 = 20y \qquad\qquad 2x + 2y\frac{dy}{dx} - 30 = 0$

$2x = \cancel{20}^{10}\frac{dy}{dx} \qquad\qquad \frac{30-2x}{2y} = \frac{dy}{dx} \Rightarrow \frac{15-x}{y}$

$\frac{dy}{dx} = \frac{x}{10}$

$(y - y_1) = \frac{-10}{x_1}(x - x_1) \rightarrow$ If passes through centre of circle.

centre $= (15, 0)$

$$-y_1 = \frac{-10}{x_1}(15-x_1)$$

$$-x_1y_1 = -150 + 10x$$

$$10x_1 - x_1y_1 + 150 = 0$$

(x_1, y_1) lies on curve $y_1 = \frac{x_1^2}{20}$

$$10x_1 - x_1 \times \frac{x_1^2}{20} + 150 = 0 \Rightarrow x_1^3 + 200x_1 - 3000 = 0$$

$$\boxed{x_1 = 10 \ , \ y = 5}$$

eqn of normal $\Rightarrow (y-5) = -1(x-10)$

$$y - 5 = -x + 10$$

$$x + y = 15$$

It cuts the circle at (x_2, y_2).

$$x_2 + y_2 = 15 \Rightarrow x_2 = 15 - y_2$$

$$(15-y_2)^2 + y_2^2 - 30(15-y_2) + 221 = 0$$

$$225 + y_2^2 - 30y_2 + y_2^2 - 450 + 30y_2 + 221 = 0$$

$$2y_2^2 - 4 = 0 \Rightarrow y_2 = \pm\sqrt{2}$$

$$x_2 = (15 \mp \sqrt{2})$$

$$\boxed{(x_2, y_2) = 15 \mp \sqrt{2}, \pm\sqrt{2}}$$

$$\rightarrow \quad (10 - 15 + \sqrt{2})^2 + \left(\frac{100}{20} - \sqrt{(17-5)(1)}\right)$$

NO. OF SOLUTION ....

If $f(x) = g(x)$ has only one solution, its mean $f(x)$ & $g(x)$ touch each other and we have to use condition of tangency.

Ex:- find the value of K for which eq^n $3x^4 = Kx - 1$ has Exactly one sol.

$f(x) = 3x^4$ $\qquad$ $g(x) = Kx - 1$

$\frac{dy}{dx} = 12x^3\Big|_{x_1} \Rightarrow 12x_1^3$ $\qquad$ $\frac{dy}{dx}\Big|_{x=x_1} = K$ $\qquad$ $\boxed{K = 12x_1^3}$

$3x_1^4 = Kx_1 - 1$

$3x_1^4 = 12x_1^4 - 1$

$1 = 9x_1^4 \Rightarrow x_1^4 = \frac{1}{9}$ $\qquad$ $x_1 = \pm\frac{1}{\sqrt{3}}$

$K = \pm 12 \times \frac{1}{3\sqrt{3}} = \pm\frac{4}{\sqrt{3}} \Rightarrow \boxed{K = \pm\frac{4}{\sqrt{3}}}$

find the value of K if $\ln x = Kx^2$ has exactly one solution.

$\rightarrow$ $f(x) = \ln x$ $\qquad$ $g(x) = Kx^2$

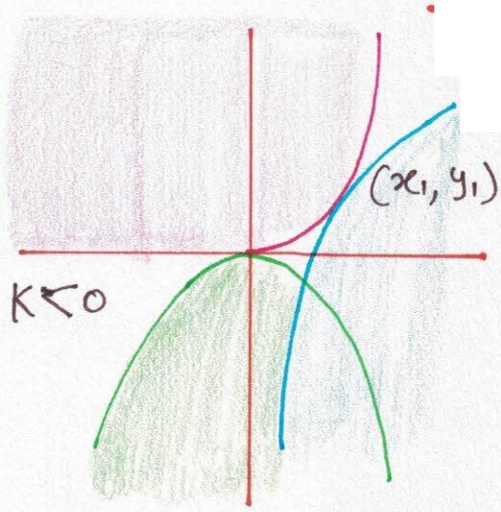
$K > 0$, $\frac{dy}{dx}\Big|_{x_1} = \frac{1}{x_1}$ $\qquad$ $\frac{dy}{dx}\Big|_{x_1} = 2Kx_1$

$\frac{1}{x_1} = 2Kx_1 \Rightarrow 2Kx_1^2 = 1 \Rightarrow x_1^2 = \frac{1}{2K} \Rightarrow K = \frac{1}{2x_1^2}$

$\ln x_1 = \frac{1}{2} \Rightarrow x_1 = e^{1/2} \Rightarrow K = \frac{1}{2e}$

$K = 0 \rightarrow$ One solution

For $K < 0$, all value satisfy



$\boxed{K \in (-\infty, 0) \cup \left\{\frac{1}{2e}\right\}}$

Question

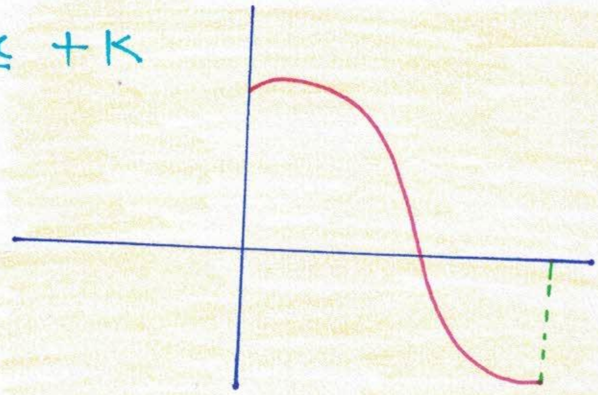
Find the value of k if $1-\cos x = \frac{\sqrt{3}}{2}|x| + k$ has exactly one solution in $(0,\pi)$.

$$1-\cos x = \frac{\sqrt{3}x}{2} + K$$

$$f(x) = 1-\cos x$$

$$\frac{dy}{dx} = 1+\sin x$$

$$= \sin x$$



$$\sin x = \frac{\sqrt{3}}{2}$$

$$x = \left(\frac{\pi}{3}, \frac{2\pi}{3}\right)$$

$$g(x) = \frac{\sqrt{3}}{2}x + K$$

$$g'(x) = \frac{\sqrt{3}}{2}$$

$$1-\cos\frac{\pi}{3} = \frac{\sqrt{3}}{2}\frac{\pi}{3} + K$$

$$\frac{1}{2} = \frac{\sqrt{3}\pi}{6} + k \Rightarrow k = \frac{1}{2} - \frac{\sqrt{3}\pi}{6} = \frac{3-\sqrt{3}\pi}{6}$$

$$1-\cos\frac{2\pi}{3} = \frac{\sqrt{3}}{2}\times\frac{2\pi}{3} + K$$

$$\frac{3}{2} - \frac{\pi}{\sqrt{3}} = K$$

$$\boxed{K = \left(\frac{3}{2} - \frac{\pi}{\sqrt{3}}, \frac{3-\sqrt{3}\pi}{6}\right)}$$